

ON THE ZERO-IN-THE-SPECTRUM CONJECTURE

M. FARBER AND S. WEINBERGER

ABSTRACT. We prove that the answer to the "zero-in-the-spectrum" conjecture, in its form, suggested by J. Lott, is negative. Namely, we show that for any $n \geq 6$ there exists a closed n -dimensional smooth manifold M^n , so that zero does not belong to the spectrum of the Laplace-Beltrami operator acting on the L^2 forms of all degrees on the universal covering \tilde{M} .

1. The Main result

M. Gromov formulated the following conjecture (cf. [G1], p. 120; [G2], p.21, Problem 0.5. F_1'' , and also [G2], p. 238):

Conjecture A. *Let M be a closed aspherical manifold; is it true that zero is always in the spectrum of the Laplace-Beltrami operator Δ_p , acting on the square integrable p -forms on the universal covering \tilde{M} , for some p ?*

If the Strong Novikov Conjecture holds for the fundamental group $\pi_1(M)$, then $0 \in \text{Spec}(\Delta_p)$ for some p , cf. [L], p. 371. Hence a counterexample to Conjecture A would be also a counterexample to the Strong Novikov Conjecture.

G. Yu obtained in [Yu1], [Yu2] results, confirming Conjecture A under some additional assumptions.

In 1991 J. Lott raised a more general "zero-in-the-spectrum" question:

Conjecture B. *Is it true, that for any complete Riemannian manifold M zero is always in the spectrum of the Laplace-Beltrami operator Δ_p , acting on the square integrable p -forms on M , for some p ?*

We refer to the survey articles [L] and [Lu].

J. Lott showed in [L] that Conjecture B is true for manifolds of low dimension and also for some classes of higher dimensional manifolds.

In this article we give negative answers to Conjecture B and also to a version of Conjecture A where one removes the assumption of asphericity of M . We prove the following Theorem.

Date: October 1, 1999.

1991 Mathematics Subject Classification. Primary 57Q10; Secondary 53C99.

Key words and phrases. Zero in the spectrum conjecture, extended L^2 cohomology.

Partially supported by the US - Israel Binational Science Foundation and by the Minkowski Center for Geometry.

Theorem 1. *For any $n \geq 6$ there exists a closed n -dimensional smooth manifold M , so that for any $p = 0, 1, \dots, n$ the zero does not belong to the spectrum of the Laplace-Beltrami operator*

$$\Delta_p : \Lambda_{(2)}^p(\tilde{M}) \rightarrow \Lambda_{(2)}^p(\tilde{M}),$$

acting on the space of L^2 -forms $\Lambda_{(2)}^p(\tilde{M})$ on the universal covering \tilde{M} of M .

Our proof of Theorem 1 will be based on the fact that it can be restated in an equivalent form using the notion of extended L^2 -homology, introduced in [F1]:

Theorem 2. *For any $n \geq 6$ there exists a closed orientable n -dimensional smooth manifold M , so that extended L^2 homology $\mathcal{H}_p(M; \ell^2(\pi)) = 0$ vanishes for all p . Here π denotes the fundamental group $\pi = \pi_1(M)$, and $\ell^2(\pi)$ denotes the L^2 -completion of the group ring $\mathbf{C}[\pi]$.*

Equivalence between Theorem 1 and Theorem 2 can be established as follows. Zero not in the spectrum of the Laplacian $\Delta_p : \Lambda_{(2)}^p(\tilde{M}) \rightarrow \Lambda_{(2)}^p(\tilde{M})$ for all p is equivalent to vanishing of the extended L^2 cohomology $\mathcal{H}^*(M; \ell^2(\pi))$, cf. [F1], according to the De Rham Theorem for extended cohomology, cf. section 7 of [F2] and also [S]. Vanishing of the extended L^2 -cohomology is equivalent to vanishing of the extended L^2 -homology $\mathcal{H}_*(M; \ell^2(\pi))$, because of the Poincaré duality, cf. [F1], Theorem 6.7.

The proof of Theorem 2 is based on the following Theorem:

Theorem 3. *There exists a finite 3-dimensional polyhedron Y with fundamental group $\pi_1(Y) = \pi = F \times F \times F$, where F denotes a free group with two generators, such that the extended L^2 -homology $\mathcal{H}_p(Y; \ell^2(\pi)) = 0$ vanishes for all $p = 0, 1, \dots$.*

The strategy of our proof of Theorems 2 and 3 is similar to the method used by M.A. Kervaire [K], who constructed smooth homology spheres with prescribed fundamental groups. Our proof uses L^2 -analog of the Hopf exact sequence.

The authors are thankful to B. Eckmann and A. Connes for helpful conversations.

2. Proofs of Theorems 2 and 3

A. Let π be a discrete group given by a finite presentation

$$\pi = \langle x_1, x_2, \dots, x_n : r_1 = 1, r_2 = 1, \dots, r_m = 1 \rangle$$

by generators and relations. We will assume that:

(a) The extended L^2 -homology of π in dimensions 0, 1 and 2 vanishes, i.e.

$$\mathcal{H}_0(\pi; \ell^2(\pi)) = \mathcal{H}_1(\pi; \ell^2(\pi)) = \mathcal{H}_2(\pi; \ell^2(\pi)) = 0.$$

(b) Let X be a finite cell complex with $\pi_1(X) = \pi$ having one zero-dimensional cell, n cells of dimension 1 and m cells of dimension two, constructed in the usual way out of the given presentation of π . Then the second homotopy group $\pi_2(X)$ of X , viewed as a $\mathbf{Z}[\pi]$ -module, is free and finitely generated.

Our purpose is to show that there exists a 3-dimensional cell complex Y , obtained from X by first taking a bouquet with finitely many copies of S^2 and then adding a finite number of 3-dimensional cells, so that

$$\mathcal{H}_i(Y; \ell^2(\pi)) = 0 \quad \text{for any } i = 0, 1, \dots \quad (1)$$

B. L^2 -Hopf exact sequence. First we will calculate the extended L^2 homology of X using the spectral sequence constructed in Theorem 9.7 of [F1]. We will work in the von Neumann category \mathcal{C}_π of Hilbert representations of π , cf. [F2], §2, example 5. We will denote by $\mathcal{E}(\mathcal{C}_\pi)$ the corresponding extended abelian category, cf. [F2], §1. Let \tilde{X} be the universal covering of X . We will use the functors

$$\mathcal{T}or_p^\pi(\ell^2(\pi), H_q(\tilde{X}; \mathbf{C}))$$

with values in the extended abelian category $\mathcal{E}(\mathcal{C}_\pi)$, which are defined in [F1], page 660 under the assumption that the homology modules $H_q(\tilde{X}; \mathbf{C})$ of the universal covering admit finite free resolutions. In our case only two of these homology modules can be nonzero (for $q = 0$ and $q = 2$), and (since $H_2(\tilde{X}; \mathbf{C}) = \mathbf{C} \otimes \pi_2(X)$) our assumption (b) guarantees this finiteness condition for $q = 2$. The functor $\mathcal{T}or_0^\pi(\ell^2(\pi), H_q(\tilde{X}; \mathbf{C}))$ can be denoted by

$$\ell^2(\pi) \tilde{\otimes}_\pi H_q(\tilde{X}; \mathbf{C}). \quad (2)$$

It is an analog of the tensor product, cf. [F2], §6. Note that in general it takes values in the extended category $\mathcal{E}(\mathcal{C}_\pi)$, i.e. it may have a nontrivial torsion part.

By Theorem 9.7 of [F1], there exists a spectral sequence in the abelian category $\mathcal{E}(\mathcal{C}_\pi)$ with the following properties:

- the initial term $E_{p,q}^2$ equals $\mathcal{T}or_p^\pi(\ell^2(\pi), H_q(\tilde{X}; \mathbf{C}))$.
- The spectral sequence converges to the extended homology $\mathcal{H}_{p+q}(X; \ell^2(\pi))$.

For $q = 0$ we have $H_0(\tilde{X}; \mathbf{C}) = \mathbf{C}$, and $\mathcal{T}or_q^\pi(\ell^2(\pi), \mathbf{C})$ can be also understood as the extended L^2 homology of the Eilenberg - MacLane space $K(\pi, 1)$. We will use notation

$$\mathcal{T}or_q^\pi(\ell^2(\pi), \mathbf{C}) = \mathcal{H}_q(\pi; \ell^2(\pi)). \quad (3)$$

It is an analog of the group homology.

Since X is two-dimensional, the spectral sequence contains only two rows ($q = 0$ and $q = 2$) and may have only one nontrivial differential. Hence we obtain the following isomorphisms:

$$\mathcal{H}_0(X; \ell^2(\pi)) \simeq \mathcal{H}_0(\pi; \ell^2(\pi)) \quad \text{and} \quad \mathcal{H}_1(X; \ell^2(\pi)) \simeq \mathcal{H}_1(\pi; \ell^2(\pi)). \quad (4)$$

These are Hurewicz type isomorphisms. The first nontrivial differential of the E^2 -term is $d_2 : E_{3,0}^2 \rightarrow E_{0,2}^2$. Here $E_{3,0}^2 = \mathcal{H}_3(\pi; \ell^2(\pi))$ and $E_{0,2}^2 = \ell^2(\pi) \tilde{\otimes}_\pi H_2(\tilde{X}; \mathbf{C})$. Using the Hurewicz isomorphism $H_2(\tilde{X}) \simeq \pi_2(\tilde{X}) \simeq \pi_2(X)$, we may write

$$E_{0,2}^2 = \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X)$$

and the above differential is

$$d_2 : \mathcal{H}_3(\pi; \ell^2(\pi)) \rightarrow \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X). \quad (5)$$

Note also that this differential must be a monomorphism (viewed as a morphism of the abelian category $\mathcal{E}(\mathcal{C}_\pi)$), since $\mathcal{H}_3(X; \ell^2(\pi)) = 0$ (recall that X is two-dimensional). The spectral sequence above yields the following exact sequence

$$0 \rightarrow \mathcal{H}_3(\pi; \ell^2(\pi)) \xrightarrow{d_2} \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X) \xrightarrow{h} \mathcal{H}_2(X; \ell^2(\pi)) \rightarrow \mathcal{H}_2(\pi; \ell^2(\pi)) \rightarrow 0. \quad (6)$$

It is an L^2 analog of the *Hopf's exact sequence*.

We conclude (using (4) and our assumptions (a)) that

$$\mathcal{H}_0(X; \ell^2(\pi)) = \mathcal{H}_1(X; \ell^2(\pi)) = 0$$

and $\mathcal{H}_2(X; \ell^2(\pi))$ can be found from the exact sequence

$$0 \rightarrow \mathcal{H}_3(\pi; \ell^2(\pi)) \rightarrow \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X) \xrightarrow{h} \mathcal{H}_2(X; \ell^2(\pi)) \rightarrow 0. \quad (7)$$

C. We will now specialize our discussion to the following group

$$\pi = F \times F \times F,$$

where F is a free group with two generators. We will denote the free generators of the factor number r (where $r = 1, 2, 3$) by a_1^r, a_2^r . We will fix the presentation of π given by 6 generators $a_1^1, a_2^1, a_1^2, a_2^2, a_1^3, a_2^3$ and the following 12 relations

$$(a_i^k, a_j^l) = 1, \quad \text{for } k \neq l, \quad k, l \in \{1, 2, 3\}, \quad i, j \in \{1, 2\},$$

where $(v, w) = vwv^{-1}w^{-1}$ denotes the commutator.

π satisfies condition (a) above, as follows from the Kunneth theorem for the extended L^2 -cohomology, cf. Appendix, Theorem 6. Here we use that $\mathcal{H}_j(F; \ell^2(F))$ is nonzero only for $j = 1$ and has no torsion; hence the terms containing the periodic product in formula (34), vanish; cf. Proposition 4, statement (b).

Let us show that this group π , together with its specified presentation, satisfies condition (b). The two-dimensional complex X constructed out of this presentation will have one zero-dimensional cell e^0 , six 1-dimensional cells e_i^1, e_i^2, e_i^3 and 12 two-dimensional cells $e_{ij}^{12}, e_{ij}^{13}, e_{ij}^{23}$. Here e_i^k denotes the 1-cell corresponding to the generator a_i^k and e_{ij}^{kl} denotes the 2-cell corresponding to the relation $(a_i^k, a_j^l) = 1$.

Let $0 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$ denote the chain complex of the universal covering \tilde{X} . The boundary homomorphism acts as follows

$$\begin{aligned} \partial e_i^k &= (a_i^k - 1)e^0, \\ \partial e_{ij}^{kl} &= (a_i^k - 1)e_j^l - (a_j^l - 1)e_i^k. \end{aligned}$$

Using the Hurewicz isomorphisms $\pi_2(X) = \pi_2(\tilde{X}) = H_2(\tilde{X})$, we may compute the group $\pi_2(X)$, viewed as a $\mathbf{Z}[\pi]$ -module, as the kernel of $\partial : C_2 \rightarrow C_1$. Let

$$x \in C_2, \quad x = \sum_{ij} \lambda_{ij}^{12} e_{ij}^{12} + \sum_{ij} \lambda_{ij}^{13} e_{ij}^{13} + \sum_{ij} \lambda_{ij}^{23} e_{ij}^{23}, \quad \lambda_{ij}^{kl} \in \mathbf{Z}[\pi],$$

be an element with $\partial x = 0$. Then the following equations hold

$$\begin{aligned}\sum_i \lambda_{ij}^{12}(a_i^1 - 1) &= \sum_i \lambda_{ji}^{23}(a_i^3 - 1), \\ \sum_j \lambda_{ij}^{12}(a_j^2 - 1) + \sum_j \lambda_{ij}^{13}(a_j^3 - 1) &= 0, \\ \sum_i \lambda_{ij}^{13}(a_i^1 - 1) + \sum_i \lambda_{ij}^{23}(a_i^2 - 1) &= 0.\end{aligned}$$

Hence we may write

$$\begin{aligned}\lambda_{ij}^{12} &= \sum_k \mu_{ijk}^{12}(a_k^3 - 1), \quad \mu_{ijk}^{12} \in \mathbf{Z}[\pi], \\ \lambda_{ij}^{23} &= \sum_k \mu_{ijk}^{23}(a_k^1 - 1), \quad \mu_{ijk}^{23} \in \mathbf{Z}[\pi], \\ \lambda_{ij}^{13} &= \sum_k \mu_{ijk}^{13}(a_k^2 - 1), \quad \mu_{ijk}^{13} \in \mathbf{Z}[\pi].\end{aligned}$$

Therefore we obtain

$$\mu_{ijk}^{12} = \mu_{jki}^{23} = -\mu_{ikj}^{13}. \quad (8)$$

Conversely, any system $\mu_{ijk}^{rs} \in \mathbf{Z}[\pi]$ satisfying (8) determines a cycle $x \in C_2$, $\partial x = 0$. This proves that $\pi_2(X)$ is a free $\mathbf{Z}[\pi]$ -module of rank 8 with the basis

$$x_{ijk} = (a_i^1 - 1)e_{jk}^{23} - (a_j^2 - 1)e_{ik}^{13} + (a_k^3 - 1)e_{ij}^{12}, \quad i, j, k \in \{1, 2\}. \quad (9)$$

Note that the Eilenberg-MacLane complex $K = K(\pi, 1)$ is $B \times B \times B$, where B is the bouquet of two circles; K is obtained from X by adding 8 three-dimensional cells e_{ijk} , where $i, j, k \in \{1, 2\}$, which correspond to different triple products of 1-dimensional cells of B . It is easy to see that the boundary of e_{ijk} is given by

$$\partial e_{ijk} = x_{ijk} \in \pi_2(X).$$

The chain complex of the universal covering \tilde{K} is $0 \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$, where C_3 is the free $\mathbf{Z}[\pi]$ -module generated by the cells e_{ijk} and the rest is the same as the chain complex of \tilde{X} .

For a discrete group π we will denote by $C_{\mathbf{R}}^*(\pi) \subset C_r^*(\pi)$ the real part of the reduced C^* -algebra, i.e. the norm closure of the real group ring $\mathbf{R}[\pi] \subset \mathbf{C}[\pi]$.

D. Proposition. *Let F be the free group with generators a_1, a_2 . Then there exist $u_1, u_2 \in C_{\mathbf{R}}^*(F) \subset C_r^*(F)$ such that*

- (i) $u_1(a_1 - 1) + u_2(a_2 - 1) = 0$,
- (ii) *for any pair $v_1, v_2 \in \ell^2(\pi)$ with*

$$v_1(a_1 - 1) + v_2(a_2 - 1) = 0 \quad (10)$$

there exists a unique $w \in \ell^2(\pi)$ such that

$$v_1 = wu_1, \quad v_2 = wu_2.$$

Here we consider F as a subgroup of $\pi = F \times F \times F$ identifying it with one of the factors. The reduced C^* -algebra $C_r^*(F) \subset C_r^*(\pi)$ acts in the usual way on $\ell^2(\pi)$.

Proof. For convenience, we will assume in the proof that F is the third factor in π . Consider the standard complex

$$\ell^2(F) \oplus \ell^2(F) \xrightarrow{d} \ell^2(F), \quad (v_1, v_2) \mapsto v_1(a_1 - 1) + v_2(a_2 - 1). \quad (11)$$

calculating the extended L^2 homology of the bouquet $S^1 \vee S^1$ of two circles with coefficients in $\ell^2(F)$. Since F is not amenable, we know from Brooks' theorem that d is an epimorphism, i.e. $\mathcal{H}_0(S^1 \vee S^1; \ell^2(F)) = 0$. The Euler characteristic calculation shows that $\ker d = \mathcal{H}_1(S^1 \vee S^1; \ell^2(F))$ is one dimensional, i.e. it is isomorphic to $\ell^2(F)$. Here we use the fact that the von Neumann algebra $\mathcal{N}(F)$ is a factor.

Let $P : \ell^2(F) \oplus \ell^2(F) \rightarrow \ell^2(F) \oplus \ell^2(F)$ be the orthogonal projection onto $\ker d$. We claim that the element $P(1, 0)$ belongs to

$$C_{\mathbf{R}}^*(F) \oplus C_{\mathbf{R}}^*(F) \subset C_r^*(F) \oplus C_r^*(F) \subset \ell^2(F) \oplus \ell^2(F). \quad (12)$$

Let d^* be the adjoint of d . Then $\ker d = \ker(d^*d)$. Moreover, the image of d^*d is closed and thus zero is an isolated point in the spectrum of d^*d . Hence we may use the holomorphic functional calculus (Cauchy's formula) in order to express the projector P as

$$P = \frac{1}{2\pi i} \int_{\Gamma} (z - d^*d)^{-1} dz,$$

where Γ is a small circle around the origin. This explains that $P(v_1, v_2)$ belongs to $C_r^*(F) \oplus C_r^*(F)$ (cf. (12)), assuming that v_1, v_2 lie in the reduced C^* -algebra $C_r^*(F)$. Moreover, since the operator d^*d is real, we obtain that $P(v_1, v_2) \in C_{\mathbf{R}}^*(F) \oplus C_{\mathbf{R}}^*(F)$, for $v_1, v_2 \in C_{\mathbf{R}}^*(F)$.

We will set now

$$(u_1, u_2) = P(1, 0).$$

Then (i) is clearly satisfied.

We want to show that the restriction of P on the first summand $\ell^2(F)$ in (11) gives an isomorphism $P : \ell^2(F) \rightarrow \ker d$. Since both $\ker d$ and $\ell^2(F)$ have von Neumann dimension one, and the spectrum of P contains only 0 and 1, we conclude that it is enough to show that $P(v, 0) = 0$ for $v \in \ell^2(F)$ implies $v = 0$. If $P(v, 0) = 0$ i.e. $(v, 0) \in (\ker d)^\perp$ then $\langle v, \ker d \rangle = 0$, i.e. v is orthogonal to the projection of $\ker d$ on the first summand $\ell^2(F)$. From this we will obtain that necessarily $v = 0$ if we will show that the projection of $\ker d$ on the first summand is dense.

Let $f_i : \ell^2(F) \rightarrow \ell^2(F)$ be operator $x \mapsto x(a_i - 1)$, where $i = 1, 2$. It is clear that f_1 and f_2 are injective and hence their images are dense. We claim that $f_1^{-1}(\text{im } f_2)$ is dense. If not, let H denote the orthogonal complement to the closure of $f_1^{-1}(\text{im } f_2)$. Then we may apply Proposition 2.4 from [F2]; it implies that H must intersect $\text{im } f_2$, which is impossible. Hence it follows that the projection of $\ker d$ on the first summand $\ell^2(F)$ (which coincides with $f_1^{-1}(\text{im } f_2)$) is dense.

As a result we obtain from the above arguments that for any pair $(v_1, v_2) \in \ker d$ (i.e. which is a solution of (10)) there exists $w \in \ell^2(F)$, so that $P(w, 0) = (v_1, v_2)$, i.e. $v_1 = wu_1$ and $v_2 = wu_2$. This is in fact a part of our statement (ii).

In order to prove (ii) in full generality, observe that

$$\ell^2(\pi) = \ell^2(F) \hat{\otimes} \ell^2(F) \hat{\otimes} \ell^2(F), \quad (13)$$

(cf. appendix) and thus (using the Kunneth theorem for extended L^2 homology, cf. Theorem 5) we find that the kernel of the operator

$$d : \ell^2(\pi) \oplus \ell^2(\pi) \rightarrow \ell^2(\pi), \quad (v_1, v_2) \mapsto v_1(a_1 - 1) + v_2(a_2 - 1),$$

equals $\ell^2(F) \hat{\otimes} \ell^2(F) \hat{\otimes} \mathcal{H}_1(S^1 \vee S^1; \ell^2(F))$. (ii) now follows. \square

E. Now we describe the kernel of the Hurewicz homomorphism

$$h : \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X) \rightarrow \mathcal{H}_2(X; \ell^2(\pi)).$$

Let $u_i^s \in C_r^*(\pi)$, where $s = 1, 2, 3$ and $i = 1, 2$, denote the element given by Proposition **D** applied to the factor $F \subset \pi$ number $s = 1, 2, 3$. Here we consider $C_r^*(F)$ as being canonically embedded into the von Neumann algebra $\mathcal{N}(\pi)$.

We claim that *the kernel of the Hurewicz homomorphism h is generated by the element*

$$y = \sum_{ijk} u_i^1 u_j^2 u_k^3 x_{ijk} \in C_{\mathbf{R}}^*(\pi) \tilde{\otimes}_\pi \pi_2(X). \quad (14)$$

More precisely, our statement is that any element $x \in \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X)$ with $h(x) = 0$ has the form $x = \mu y$ for some $\mu \in \ell^2(\pi)$.

Note that the product μy has sense because the coefficients of y in the basis x_{ijk} belong to $C_{\mathbf{R}}^*(\pi) \subset C_r^*(\pi)$.

First, it is easy to check (using (9)) that $h(y) = 0$.

Let

$$x = \sum_{ijk} \mu_{ijk} x_{ijk} \in \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X), \quad h(x) = 0,$$

be an arbitrary element of $\ker h$, where $\mu_{ijk} \in \ell^2(\pi)$. Using (9), we obtain (equating to zero the coefficients of the cells e_{jk}^{23}) that for any pair of indices j, k holds

$$\sum_{i=1}^2 \mu_{ijk} (a_i^1 - 1) = 0.$$

Hence, applying Proposition **D**, we conclude that there exist $\mu_{jk} \in \ell^2(\pi)$ such that

$$\mu_{ijk} = \mu_{jk} u_i^1. \quad (15)$$

We write again $h(x) = 0$, equating to zero the coefficients of the cells e_{ik}^{13} and using (15). We obtain that for any pair of indices i, k holds

$$\sum_j \mu_{jk} u_i^1 (a_j^2 - 1) = \left[\sum_j \mu_{jk} (a_j^2 - 1) \right] u_i^1 = 0. \quad (16)$$

Note that $wu_1^s = 0$ for $w \in \ell^2(\pi)$ implies $wu_2^s = 0$ (using (10)) and from the uniqueness statement in Proposition **D**, (ii), we obtain that $w = 0$. Therefore (16) implies

$$\sum_j \mu_{jk} (a_j^2 - 1) = 0$$

and hence using Proposition **D**,

$$\mu_{jk} = \mu_k u_j^2, \quad \text{where } \mu_k \in \ell^2(\pi).$$

Substitute again $\mu_{ijk} = \mu_k u_i^1 u_j^2$ into $h(x) = 0$ and equating to zero the coefficients of the cells e_{ik}^{13} we obtain

$$\left[\sum_k \mu_k (a_k^3 - 1) \right] u_i^1 u_j^2 = 0, \quad \text{and hence } \sum_k \mu_k (a_k^3 - 1) = 0. \quad (17)$$

Using Proposition **D** as above we finally obtain

$$\mu_k = \mu u_k^3, \quad \text{where } \mu \in \ell^2(\pi).$$

Therefore, we find that $\mu_{ijk} = \mu u_i^1 u_j^2 u_k^3$ and $x = \mu y$. \square

F. Our goal is to show that *one may add 8 cells of dimension 3 to the bouquet $X \vee S^2$ such that the obtained 3-dimensional complex Y will have all trivial extended L^2 homology*

$$\mathcal{H}_j(Y; \ell^2(\pi)) = 0, \quad j = 0, 1, \dots$$

For the proof, let's examine again the exact sequence (7):

$$0 \rightarrow \mathcal{H}_3(\pi; \ell^2(\pi)) \xrightarrow{\phi} \ell^2(\pi) \tilde{\otimes}_{\pi} \pi_2(X) \xrightarrow{h} \mathcal{H}_2(X; \ell^2(\pi)) \rightarrow 0. \quad (18)$$

As we know, ϕ maps the generator y of $\mathcal{H}_3(\pi; \ell^2(\pi))$ according to formula (14), i.e. ϕ is given by a matrix with entries in $C_{\mathbf{R}}^*(\pi) \subset C_r^*(\pi)$. Let

$$Q : \ell^2(\pi) \tilde{\otimes}_{\pi} \pi_2(X) \rightarrow \ell^2(\pi) \tilde{\otimes}_{\pi} \pi_2(X)$$

denote the orthogonal projection onto $(\text{im } \phi)^\perp$, the orthogonal complement of the image of ϕ . Since X is two-dimensional, $\mathcal{H}_2(X; \ell^2(\pi))$ has no torsion and therefore $\text{im } \phi$ is closed. Note that $(\text{im } \phi)^\perp$ coincides with $\ker(\phi\phi^*)$. Since the image of $\phi\phi^*$ is closed we conclude that zero is an isolated point in the spectrum of $\phi\phi^*$ and hence we may write

$$Q = \frac{1}{2\pi i} \int_{\Gamma} (z - \phi\phi^*)^{-1} dz,$$

where Γ is a small circle round zero. Therefore, *in the basis x_{ijk} the projector Q is given a (8×8) -matrix with entries in $C_{\mathbf{R}}^*(\pi)$.*

The projective $C_{\mathbf{R}}^*(\pi)$ -module determined by Q is stably free; we know that adding a free one-dimensional module (generated by y) makes it free. Therefore we may consider the bouquet $X_1 = X \vee S^2$ so that $\mathcal{H}_2(X_1; \ell^2(\pi)) = \mathcal{H}_2(X; \ell^2(\pi)) \oplus \ell^2(\pi)$ and $\pi_2(X_1) = \pi_2(X) \oplus \mathbf{Z}[\pi]$. Thus, the exact sequence (18) for X_1

$$0 \rightarrow \mathcal{H}_3(\pi; \ell^2(\pi)) \xrightarrow{\psi} \ell^2(\pi) \tilde{\otimes}_{\pi} \pi_2(X_1) \xrightarrow{h} \mathcal{H}_2(X_1; \ell^2(\pi)) \rightarrow 0. \quad (19)$$

will have the following property: *the orthogonal projection*

$$Q_1 : \ell^2(\pi) \tilde{\otimes}_{\pi} \pi_2(X_1) \rightarrow \ell^2(\pi) \tilde{\otimes}_{\pi} \pi_2(X_1)$$

onto $(\text{im } \psi)^{\perp}$ is given by a (9×9) -matrix with entries in $C_{\mathbf{R}}^(\pi)$ which determines a free $C_{\mathbf{R}}^*(\pi)$ -module of rank 8.*

We may reformulate the last statement as follows: there exists a $\mathbf{Z}[\pi]$ -homomorphism

$$\gamma : (\mathbf{Z}[\pi])^8 \rightarrow C_{\mathbf{R}}^*(\pi) \tilde{\otimes}_{\pi} \pi_2(X_1) \quad (20)$$

such that the following composite

$$\ell^2(\pi) \tilde{\otimes}_{\pi} (\mathbf{Z}[\pi])^8 \xrightarrow{1 \otimes \gamma} \ell^2(\pi) \tilde{\otimes}_{\pi} \pi_2(X_1) \xrightarrow{h} \mathcal{H}_2(X_1; \ell^2(\pi)) \quad (21)$$

is an isomorphism. Now we will use the fact that the rational group ring $\mathbf{Q}[\pi]$ is dense in $C_{\mathbf{R}}^*(\pi)$ with respect to the operator norm topology. Hence we may approximate γ by a $\mathbf{Z}[\pi]$ -homomorphism

$$\gamma_1 : (\mathbf{Z}[\pi])^8 \rightarrow \mathbf{Q}[\pi] \otimes_{\pi} \pi_2(X_1)$$

so that the similar composition (21) is an isomorphism. Finally, we may multiply γ_1 by a large integer N to obtain a $\mathbf{Z}[\pi]$ -homomorphism

$$\gamma_2 : (\mathbf{Z}[\pi])^8 \rightarrow \mathbf{Z}[\pi] \otimes_{\pi} \pi_2(X_1) = \pi_2(X_1)$$

such that the composition

$$(\ell^2(\pi))^8 = \ell^2(\pi) \tilde{\otimes}_{\pi} (\mathbf{Z}[\pi])^8 \xrightarrow{1 \otimes \gamma_2} \ell^2(\pi) \tilde{\otimes}_{\pi} \pi_2(X_1) \xrightarrow{h} \mathcal{H}_2(X_1; \ell^2(\pi)) \quad (22)$$

is an isomorphism.

Let $z_1, \dots, z_8 \in \pi_2(X_1)$ be images of a free basis of $(\mathbf{Z}[\pi])^8$ under γ_2 . Realize each z_j by a continuous map $f_j : S^2 \rightarrow X_1$, where $j = 1, \dots, 8$, and let

$$Y = X_1 \cup e_1^3 \cup \dots \cup e_8^3$$

be obtained from X_1 by glueing 8 three-dimensional cells to X_1 along f_1, \dots, f_8 . We claim that

$$\mathcal{H}_j(Y; \ell^2(\pi)) = 0 \quad \text{for all } j = 0, 1, \dots. \quad (23)$$

In order to show this, we note that $\mathcal{H}_j(Y, X; \ell^2(\pi))$ vanishes for all $j \neq 3$ and the 3-dimensional extended L^2 homology $\mathcal{H}_3(Y, X; \ell^2(\pi))$ equals $(\ell^2(\pi))^8$. The boundary homomorphism $\partial : \mathcal{H}_3(Y, X; \ell^2(\pi)) \rightarrow \mathcal{H}_2(X; \ell^2(\pi))$ is an isomorphism since it coincides

with (22). Hence (23) follows from the homological exact sequence of the pair (Y, X) . This completes the proof of Theorem 3.

G. Now we may complete the proof of Theorem 2. We have constructed above a finite 3-dimensional polyhedron Y . For any $n \geq 6$ we may embed Y into \mathbf{R}^{n+1} as a subpolyhedron. Let $N \subset \mathbf{R}^{n+1}$ be the regular neighborhood of $Y \subset \mathbf{R}^{n+1}$. We will define M as the boundary of N , i.e. $M = \partial N$.

First note that the inclusion $M \rightarrow N$ induces an isomorphism of the fundamental groups and thus $\pi_1(M) = \pi = F \times F \times F$, where F is a free group in two generators. We want to show that

$$\mathcal{H}_j(M; \ell^2(\pi)) = 0, \quad \text{for all } j = 0, 1, \dots \quad (24)$$

In the exact homological sequence

$$\dots \rightarrow \mathcal{H}_j(M; \ell^2(\pi)) \rightarrow \mathcal{H}_j(N; \ell^2(\pi)) \rightarrow \mathcal{H}_j(N, M; \ell^2(\pi)) \rightarrow \dots$$

we have $\mathcal{H}_j(N; \ell^2(\pi)) = 0$. Also, $\mathcal{H}_j(N, M; \ell^2(\pi)) \simeq \mathcal{H}^{n+1-j}(N; \ell^2(\pi))$ by the Poincaré duality (cf. [F1]) and $\mathcal{H}^{n+1-j}(N; \ell^2(\pi)) = 0$ because of (23) using the Universal Coefficients Theorem (cf. [F1]). Hence, (24) follows. \square

Appendix: Kunneth theorem for extended L^2 cohomology

1. A *Hilbert category* \mathcal{C} is defined as an additive subcategory of the category of Hilbert spaces and bounded linear maps, such for any morphism $f : H \rightarrow H'$ of \mathcal{C} the inclusion $\ker(f) \subset H$ belongs to \mathcal{C} and also the adjoint map $f^* : H' \rightarrow H$ belongs to \mathcal{C} , cf. [F1]. It is shown in [F1] that any Hilbert category can be canonical embedding into an abelian category $\mathcal{E}(\mathcal{C})$, called *the extended abelian category*.

Let \mathcal{C} , \mathcal{C}' and \mathcal{C}'' be three Hilbert categories and let

$$\hat{\otimes} : \mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{C}'' \quad (25)$$

be a covariant functor of two variables (*the "tensor product"*) such that

- (a) for $H \in \text{Ob}(\mathcal{C})$ and $H' \in \text{Ob}(\mathcal{C}')$ the image $H \hat{\otimes} H'$ has as the underlying Hilbert space the tensor product of Hilbert spaces H and H' ;
- (b) if $f : H \rightarrow H_1$ is a morphism of \mathcal{C} and $f' : H' \rightarrow H'_1$ is a morphism of \mathcal{C}' then $f \hat{\otimes} f' : H \hat{\otimes} H' \rightarrow H_1 \hat{\otimes} H'_1$ is the tensor product of bounded linear maps f and f' .

Recall that the tensor product of Hilbert spaces $H \hat{\otimes} H'$ is defined as the Hilbert space completion of the algebraic tensor product $H \otimes H'$ with respect to the following scalar product $\langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle \cdot \langle w, w' \rangle$.

Suppose that (C, d) and (C', d) are chain complexes in \mathcal{C} and \mathcal{C}' correspondingly. We assume that all chain complexes are graded by non-negative integers and have a finite length. Their tensor product $(C, d) \hat{\otimes} (C', d)$ (defined in the usual way) is a chain complex in \mathcal{C}'' . $(C, d) \hat{\otimes} (C', d)$ is a projective chain complex in the abelian category $\mathcal{E}(\mathcal{C}'')$ and its extended homology $\mathcal{H}_*(C \hat{\otimes} C')$ is an object of the extended category

$\mathcal{E}(\mathcal{C}'')$. Our purpose is to express the extended homology of $(C, d) \hat{\otimes} (C', d)$ in terms of the extended homology $\mathcal{H}_*(C)$ of (C, d) and $\mathcal{H}_*(C')$ of (C', d) .

2. Example. Suppose that G and H are discrete groups. Let \mathcal{C}_G denote the category of Hilbert representations of G . Recall, that an object of \mathcal{C} is a Hilbert space with a unitary G -action which can be continuously and G -equivariantly imbedded into a finite direct sum $\ell^2(G) \oplus \cdots \oplus \ell^2(G)$; morphisms of \mathcal{C} are bounded linear maps commuting with the G -action. Then we have the tensor product functor

$$\hat{\otimes} : \mathcal{C}_G \times \mathcal{C}_H \rightarrow \mathcal{C}_{G \times H} \quad (26)$$

which is of a primary interest for us.

3. Tensor and periodic products. Given a tensor product (25), it defines two bifunctors $\mathcal{E}(\mathcal{C}) \times \mathcal{E}(\mathcal{C}) \rightarrow \mathcal{E}(\mathcal{C})$, which we now describe. Let $\mathcal{X} = (\alpha : A' \rightarrow A) \in \text{Ob}(\mathcal{E}(\mathcal{C}))$ and $\mathcal{Y} = (\beta : B' \rightarrow B) \in \text{Ob}(\mathcal{E}(\mathcal{C}'))$ be two objects with α and β injective. Consider the following chain complex in \mathcal{C}''

$$0 \rightarrow A' \hat{\otimes} B' \xrightarrow{\begin{pmatrix} -1 \hat{\otimes} \beta \\ \alpha \hat{\otimes} 1 \end{pmatrix}} (A' \hat{\otimes} B) \oplus (A \hat{\otimes} B') \xrightarrow{(\alpha \hat{\otimes} 1, 1 \hat{\otimes} \beta)} A \hat{\otimes} B \rightarrow 0. \quad (27)$$

In other words, we view the objects \mathcal{X} and \mathcal{Y} as chain complexes of length 1 and then (27) is the tensor product of these chain complexes. The extended homology of (27) in dimension 0 will be called *the tensor product of \mathcal{X} and \mathcal{Y}* :

$$\mathcal{X} \hat{\otimes} \mathcal{Y} = ((\alpha \hat{\otimes} 1, 1 \hat{\otimes} \beta) : (A' \hat{\otimes} B) \oplus (A \hat{\otimes} B') \rightarrow A \hat{\otimes} B). \quad (28)$$

The extended homology of (27) in dimension 1 will be called *the periodic product of \mathcal{X} and \mathcal{Y}* :

$$\mathcal{X} * \mathcal{Y} = \left(\begin{pmatrix} -1 \hat{\otimes} \beta \\ \alpha \hat{\otimes} 1 \end{pmatrix} : A' \hat{\otimes} B' \rightarrow Z \right), \quad (29)$$

where

$$Z = \ker \left[\begin{pmatrix} -1 \hat{\otimes} \beta \\ \alpha \hat{\otimes} 1 \end{pmatrix} : (A' \hat{\otimes} B) \oplus (A \hat{\otimes} B') \rightarrow A \hat{\otimes} B \right]. \quad (30)$$

It is easy to see that $\mathcal{X} \hat{\otimes} \mathcal{Y}$ and $\mathcal{X} * \mathcal{Y}$ are covariant functors of two variables.

Proposition 4. *Let $\hat{\otimes} : \mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{C}''$ be a tensor product functor (25). Let $\mathcal{X} \in \text{Ob}(\mathcal{E}(\mathcal{C}))$ and $\mathcal{Y} \in \text{Ob}(\mathcal{E}(\mathcal{C}'))$. Then*

- (a) $\mathcal{X} \hat{\otimes} \mathcal{Y}$ is projective if both \mathcal{X} and \mathcal{Y} are projective;
- (b) $\mathcal{X} * \mathcal{Y} = 0$ if \mathcal{X} or \mathcal{Y} is projective;
- (c) $\mathcal{X} \hat{\otimes} \mathcal{Y}$ is torsion if \mathcal{X} or \mathcal{Y} is torsion;
- (d) If \mathcal{C}'' is a finite von Neumann category then $\mathcal{X} * \mathcal{Y}$ is torsion for any \mathcal{X} and \mathcal{Y} .

Proof. Statements (a) and (b) follow directly from the definitions.

Let's prove (c) assuming that $\mathcal{X} = (\alpha : A' \rightarrow A)$ is torsion, i.e. $\text{im } \alpha \subset A$ is dense. From the definition of the tensor product $\hat{\otimes}$ it follows that then the image of $\alpha \hat{\otimes} 1 : A' \hat{\otimes} B \rightarrow A \hat{\otimes} B$ is dense and hence from (28) we see that $\mathcal{X} \hat{\otimes} \mathcal{Y}$ is torsion.

It is enough to prove (d), assuming that both \mathcal{X} and \mathcal{Y} are torsion. Let $\mathcal{X} = (\alpha : A' \rightarrow A)$ and $\mathcal{Y} = (\beta : B' \rightarrow B)$ with α and β injective and having dense images. Then A' is isomorphic to A and B' is isomorphic to B (cf. [F2], §2). Therefore (d) will follow if we can show that Z (given by (30)) is isomorphic to $A \hat{\otimes} B$. The projection of Z on the first coordinate gives a morphism $Z \rightarrow A' \hat{\otimes} B$ which is injective (obviously) and has a dense image (this follows from Proposition in §2 of [F2]). Hence we obtain (using Lemma in §2 of [F2]) that Z is isomorphic to $A' \hat{\otimes} B \simeq A \hat{\otimes} B$. \square

Theorem 5 (Kunneth formula). *Extended homology $\mathcal{H}_*(C \hat{\otimes} C')$ of a tensor product, where (C, d) is a chain complex in \mathcal{C} and (C', d) is a chain complex in \mathcal{C}' , equals*

$$\mathcal{H}_n(C \hat{\otimes} C') = \bigoplus_{i+j=n} \mathcal{H}_i(C) \hat{\otimes} \mathcal{H}_j(C') \oplus \bigoplus_{i+j=n-1} \mathcal{H}_i(C) * \mathcal{H}_j(C'). \quad (31)$$

Proof. Let $Z_i \subset C_i$ and $Z'_i \subset C'_i$ denote the subspaces of cycles.

We have the decomposition $C_i = Z_i \oplus Z_i^\perp$; the boundary homomorphism vanishes on Z_i and maps Z_i^\perp into Z_{i-1} . Let's denote by D_i the short chain complex $D_i = (d : Z_{i+1}^\perp \rightarrow Z_i)$, where Z_i stands in degree i and Z_{i+1}^\perp stands in degree $i+1$. Then $C \simeq \bigoplus_{i=0}^\infty D_i$, i.e. C is isomorphic to the direct sum of the chain complexes D_i .

Similarly, we define chain complexes $D'_j = (d : Z'_{j+1}^\perp \rightarrow Z'_j)$ and $C' \simeq \bigoplus_{j=0}^\infty D'_j$. Hence we obtain

$$C \hat{\otimes} C' \simeq \bigoplus_{i,j} (D_i \hat{\otimes} D'_j), \quad \mathcal{H}_n(C \hat{\otimes} C') = \bigoplus_{i,j} \mathcal{H}_n(D_i \hat{\otimes} D'_j). \quad (32)$$

Now we observe that D_i has nontrivial homology only in dimension i and $\mathcal{H}_i(D_i) = \mathcal{H}_i(C)$; similarly, D'_j has nontrivial homology only in dimension j and $\mathcal{H}_j(D'_j) = \mathcal{H}_j(C')$. Therefore $D_i \hat{\otimes} D'_j$ has nontrivial homology only in dimensions $i+j$ and $i+j+1$, and

$$\mathcal{H}_{i+j}(D_i \hat{\otimes} D'_j) = \mathcal{H}_i(C) \hat{\otimes} \mathcal{H}_j(C'), \quad \mathcal{H}_{i+j+1}(D_i \hat{\otimes} D'_j) = \mathcal{H}_i(C) * \mathcal{H}_j(C') \quad (33)$$

according to our definition of the tensor and periodic products. Formula (31) now follows by combining (32) and (33). \square

Theorem 6 (Kunneth formula for extended L^2 homology). *Let X, X' be finite cell complexes with $\pi = \pi_1(X)$, $\pi' = \pi_1(X')$. Then*

$$\begin{aligned} \mathcal{H}_n(X \times X'; \ell^2(\pi \times \pi')) &\simeq \\ &\bigoplus_{i+j=n} \mathcal{H}_i(X; \ell^2(\pi)) \hat{\otimes} \mathcal{H}_j(X'; \ell^2(\pi')) \oplus \\ &\bigoplus_{i+j=n-1} \mathcal{H}_i(X; \ell^2(\pi)) * \mathcal{H}_j(X'; \ell^2(\pi')), \end{aligned} \quad (34)$$

where the tensor and periodic products are understood with respect to functor (26).

Proof. Let $C_*(\tilde{X})$ and $C_*(\tilde{X}')$ be the cell chain complexes of the universal coverings \tilde{X} and \tilde{X}' . We apply the previous Theorem to chain complexes $C = \ell^2(\pi) \tilde{\otimes}_{\pi} C_*(\tilde{X})$ and $C' = \ell^2(\pi') \tilde{\otimes}_{\pi'} C_*(\tilde{X}')$. Note that C is a chain complex in category \mathcal{C}_{π} (cf. Example above) and $\mathcal{H}_n(C) = \mathcal{H}_n(X; \ell^2(\pi))$. Similarly C' is a chain complex in $\mathcal{C}_{\pi'}$ and $\mathcal{H}_n(C') = \mathcal{H}_n(X'; \ell^2(\pi'))$. Formula (34) follows from (31) using the isomorphism $\ell^2(\pi) \hat{\otimes} \ell^2(\pi') = \ell^2(\pi \times \pi')$ and the fact that the chain complex $C_*(\tilde{X}) \otimes_{\mathbf{Z}} C_*(\tilde{X}')$ over $\mathbf{Z}[\pi \times \pi']$ is isomorphic to $C_*(\widetilde{X \times X'})$, where we consider the obvious product cell structure on $X \times X'$. \square

REFERENCES

- [G1] M. Gromov, *Large Riemannian Manifolds*, in: "Curvature and Topology of Riemannian manifolds", ed. K. Shiohama, T. Sakai, T. Sunada, Lecture Notes in Math., vol. 1201, 1986, pp. 108 - 121
- [G2] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, G. Niblo and M. Roller eds., London Math. Soc. Lecture Notes 182, vol. 2, 1993
- [F1] M. Farber, *Homological algebra of Novikov-Shubin invariants and Morse inequalities*, GAFA **6**(1996), 628 - 665
- [F2] M. Farber, *Von Neumann categories and extended L^2 cohomology*, K-theory, **15**(1998), 347 - 405
- [K] M.A. Kervaire, *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc. **144**(1969), 67 - 72
- [L] J. Lott, *The zero-in-the-spectrum question*, L'Enseignement Mathématique **42**(1996), 341 - 376
- [Lu] W. Lück, *L^2 -invariants of regular coverings of compact manifolds and CW-complexes*, in: Handbook of Geometric Topology, Elsevier, to appear
- [S] M. A. Shubin, *De Rham theorem for extended L^2 -cohomology*, in: "Voronezh Winter Mathematical Schools: Dedicated to Selim Krein", P. Kuchment and V. Lin editors, AMS, Advances in the Mathematical Sciences, 1998, pp. 217 - 232
- [Yu1] G. Yu, *Zero-in-the-spectrum conjecture, positive scalar curvature and asymptotic dimension*, Invent. Math., **127**(1997), pp. 99 - 126
- [Yu2] G. Yu, *The Novikov conjecture for groups with finite asymptotic dimension*, Ann. of Math. **147**(1998), pp. 325 - 355

DEPARTMENT OF MATHEMATICS, TEL AVIV UNIVERSITY, TEL AVIV, 69978, ISRAEL
E-mail address: `farber@math.tau.ac.il`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL
E-mail address: `shmuel@math.uchicago.edu`